

## ALBANESE MAP OF MODULI OF STABLE SHEAVES ON ABELIAN SURFACES

KÔTA YOSHIOKA

## 0. INTRODUCTION

Let  $X$  be a smooth projective surface defined over  $\mathbb{C}$  and  $H$  an ample line bundle on  $X$ . If  $K_X$  is trivial, Mukai [M3] introduced a quite useful notion called Mukai lattice  $(H^{ev}(X, \mathbb{Z}), \langle \ , \ \rangle)$ , where  $H^{ev}(X, \mathbb{Z}) = \oplus_i H^{2i}(X, \mathbb{Z})$ . For a coherent sheaf  $E$  on  $X$ , we can attach an element of  $H^{ev}(X, \mathbb{Z})$  called Mukai vector  $v(E) := \text{ch}(E)\sqrt{\text{td}_X}$ , where  $\text{td}_X$  is the Todd class of  $X$ . We denote the moduli space of stable sheaves  $E$  of  $v(E) = v$  by  $M_H(v)$ . If  $H$  is general (i.e. it does not lie on walls [Y1]) and  $v$  is primitive, then  $M_H(v)$  is a smooth projective scheme.

If  $X$  is a K3 surface and  $v$  is primitive, then  $M_H(v)$  is extensively studied by many authors. In particular, in many cases,  $M_H(v)$  is an irreducible symplectic manifold and the period of  $M_H(v)$  is written down in terms of Mukai lattice ([Mu3,5], [O], [Y4]).

In this paper, we shall treat the case where  $X$  is an abelian surface. In [Y2], we studied  $H^i(M_H(v), \mathbb{Z})$   $i = 1, 2$  under some assumptions on  $v$ . We also constructed a morphism  $\mathbf{a} : M_H(v) \rightarrow X \times \hat{X}$  and proved that  $\mathbf{a}$  is an albanese map for  $\langle v^2 \rangle \geq 2$ , where  $\hat{X}$  is the dual of  $X$ . In this paper, we shall consider the fiber of albanese map under the same assumptions on  $v$  in [Y2] (cf. Theorem 0.1). If  $\langle v^2 \rangle = 0$ , then Mukai showed that  $M_H(v)$  is an abelian surface (see [Mu5, (5.13)]). In this case,  $\mathbf{a}$  is an immersion. If  $\langle v^2 \rangle = 2$ , then Mukai [Mu1] and the author [Y2, Prop. 4.2] showed that  $\mathbf{a} : M_H(v) \rightarrow X \times \hat{X}$  is an isomorphism. Hence we assume that  $\langle v^2 \rangle \geq 4$ . Let  $K_H(v)$  be a fiber of  $\mathbf{a}$ . Then  $\dim K_H(v) = \langle v^2 \rangle - 2$ . Hence if  $\langle v^2 \rangle \geq 6$ , then  $\dim K_H(v) \geq 4$ . In this case, we get the following, which is an analogous result to that for a K3 surface.

**Theorem 0.1.** *Let  $X$  be an abelian surface. Let  $v = r + \xi + a\omega \in H^{ev}(X, \mathbb{Z})$ ,  $\xi \in H^2(X, \mathbb{Z})$  be a Mukai vector such that  $r > 0$ ,  $r + \xi$  is primitive and  $\langle v^2 \rangle \geq 6$ , where  $\omega$  is the fundamental class of  $X$ . Then for a general ample line bundle  $H$ ,  $K_H(v)$  is an irreducible symplectic manifold and*

$$\theta_v : v^\perp \rightarrow H^2(K_H(v), \mathbb{Z}) \quad (0.1)$$

*is an isometry of Hodge structures.*

For the definition of  $\theta_v$ , see preliminaries. Our theorem shows that Mukai lattice for an abelian surface is as important as that for a K3 surface. As an application of this theorem, we shall show that for some  $v$ ,  $M_H(v)$  is not birationally equivalent to  $\hat{Y} \times \text{Hilb}_Y^n$  for any  $Y$  (Example 1).

In section 1, we collect some known facts which will be used in this paper. Since the canonical bundle of  $M_H(v)$  is trivial,  $M_H(v)$  has a Bogomolov decomposition. We shall also construct a decomposition which will become a Bogomolov decomposition for  $M_H(v)$ .

In section 2, we shall prove Theorem 0.1. We shall first treat rank 1 case. In this case,  $K_H(v)$  is the generalized Kummer variety  $K_{n-1}$  constructed by Beauville [B], where  $\langle v^2 \rangle / 2 = n$ . Hence Theorem 0.1 follows from Beauville's description of  $H^2(K_{n-1}, \mathbb{Q})$  and some computations. For higher rank cases, we shall use the same method as in [Y2]. More precisely, we shall first treat the case where  $X$  is a product of two elliptic curves. In this case, we constructed a family of stable sheaves  $E$  of  $v(E) = v$  in [Y2]. Then it induces a birational map  $\hat{X} \times \text{Hilb}_X^n \cdots \rightarrow M_H(v)$ , and hence we get a birational map from the generalized Kummer variety  $K_{n-1}$  to  $K_H(v)$ , where  $n = \langle v^2 \rangle / 2$ . By the description of  $\theta_v(x), x \in v^\perp$  in [Y2], we get our theorem for this case. For general cases, we shall use deformation arguments as in [G-H], [O] and [Y2].

In section 3, we shall treat the remaining case. In this case, we shall prove that  $K_H(v)$  is isomorphic to a moduli space of stable sheaves on the Kummer surface associated to  $X$ .

In appendix, we shall explain a more sophisticated method to prove Theorem 0.1 at least for  $r \neq 2$ . In the K3 cases, we know that isometries of Mukai lattice are quite useful to compute period of moduli spaces [Mu3], [Y4]. Hence it is also important to study isometries in our cases. Fourier-Mukai transforms are good examples of isometries [Mu4]. Thanks to recent results of Bridgeland [Br] on Fourier-Mukai transforms, we

can replace our computations in [Y2] to a simple calculation (Proposition 4.3) for  $\geq 3$ . Thus we get another proof of our theorem for  $r \neq 2$ .

## 1. PRELIMINARIES

*Notation.*

Let  $M$  be a complex manifold. For a cohomology class  $x \in H^*(M, \mathbb{Z})$ ,  $[x]_i \in H^{2i}(X, \mathbb{Z})$  denotes the  $2i$ -th component of  $x$ .

Let  $p : X \rightarrow \text{Spec}(\mathbb{C})$  be an abelian surface or a K3 surface over  $\mathbb{C}$ . We denote the projection  $S \times X \rightarrow S$  by  $p_S$ . In this paper, we identify a divisor class  $D$  with associated line bundle  $\mathcal{O}_X(D)$ .

**1.1. Mukai lattice.** We shall recall the Mukai lattice [Mu3].

**Definition 1.1.** We define a symmetric bilinear form on  $H^{ev}(X, \mathbb{Z}) := \oplus_i H^{2i}(X, \mathbb{Z})$ :

$$\begin{aligned} \langle x, y \rangle &:= - \int_X (x^\vee y) \\ &= \int_X (x_1 y_1 - x_0 y_2 - x_2 y_0) \end{aligned}$$

where  $x = x_0 + x_1 + x_2$ ,  $x_i \in H^{2i}(X, \mathbb{Z})$  (resp.  $y = y_0 + y_1 + y_2$ ,  $y_i \in H^{2i}(X, \mathbb{Z})$ ) and  $\vee : H^{ev}(X, \mathbb{Z}) \rightarrow H^{ev}(X, \mathbb{Z})$  be the homomorphism sending  $x$  to  $x_0 - x_1 + x_2 \in H^{ev}(X, \mathbb{Z})$ . We call this lattice Mukai lattice.

For a coherent sheaf  $E$  on  $X$ ,

$$\begin{aligned} v(E) &:= \text{ch}(E) \sqrt{\text{td}_X} \\ &= \text{ch}(E)(1 + \varepsilon \omega) \end{aligned}$$

is the Mukai vector of  $E$ , where  $\omega$  is the fundamental class of  $X$  and  $\varepsilon = 0, 1$  according as  $X$  is of type abelian or K3. Then Riemann-Roch theorem is written as follows:

$$\chi(E, F) = -\langle v(E), v(F) \rangle, \quad (1.1)$$

where  $E$  and  $F$  are coherent sheaves on  $X$ .

For an abelian surface  $X$ , Mukai lattice has a decomposition

$$\begin{aligned} H^{ev}(X, \mathbb{Z}) &= H^2(X, \mathbb{Z}) \oplus H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \\ &= U^{\oplus 4}, \end{aligned} \quad (1.2)$$

where  $U$  is the hyperbolic lattice.

**1.2. Moduli of stable sheaves.** Let  $X$  be an abelian or a K3 surface, and  $H$  an ample line bundles on  $X$ . For  $v \in H^{ev}(X, \mathbb{Z})$ , let  $M_H(v)$  be the moduli of stable sheaves of Mukai vector  $v$ . By Mukai [Mu2],  $M_H(v)$  is smooth of dimension  $\langle v^2 \rangle + 2$  and has a symplectic structure. We set

$$v^\perp := \{x \in H^{ev}(X, \mathbb{Z}) \mid \langle v, x \rangle = 0\}.$$

Let  $\theta_v : v^\perp \rightarrow H^2(M_H(v), \mathbb{Z})$  be the homomorphism such that

$$\theta_v(x) := -\frac{1}{\rho} \left[ p_{M_H(v)*}((\text{ch } \mathcal{E}) \sqrt{\text{td}_X} x^\vee) \right]_1 \quad (1.3)$$

where  $\mathcal{E}$  is a quasi-universal family of similitude  $\rho$ . For a line bundle  $L$  on  $X$ , let  $T_L : H^{ev}(X, \mathbb{Z}) \rightarrow H^{ev}(X, \mathbb{Z})$  be the homomorphism sending  $x$  to  $x \text{ch}(L)$ . Then  $T_L$  is an isometry of Mukai lattice and satisfies that

$$\theta_{T_L(v)}(T_L(x)) = \theta_v(x), \quad (1.4)$$

for  $x \in v^\perp$ .

From now on, we assume that  $X$  is an abelian surface. Let  $\widehat{X}$  be the dual abelian variety of  $X$  and  $\mathcal{P}$  the Poincaré line bundle on  $\widehat{X} \times X$ . For an element  $E_0 \in M_H(v)$ , let  $\alpha : M_H(v) \rightarrow X$  be the morphism sending  $E \in M_H(v)$  to  $\det p_{\widehat{X}!}((E - E_0) \otimes (\mathcal{P} - \mathcal{O}_{\widehat{X} \times X})) \in \text{Pic}^0(\widehat{X}) = X$ , and  $\det : M_H(v) \rightarrow \widehat{X}$  the morphism sending  $E$  to  $\det E \otimes \det E_0^\vee \in \widehat{X}$ . We set  $\mathbf{a} := \alpha \times \det$ . Then the following hold [Y2, Thm. 3.1, 3.6].

**Theorem 1.1.** *Let  $v = r + \xi + a\omega$ ,  $\xi \in H^2(X, \mathbb{Z})$  be a Mukai vector such that  $r > 0$  and  $r + \xi$  is primitive. We assume that  $\dim M_H(v) = \langle v^2 \rangle + 2 \geq 6$ . Then for a general ample line bundle  $H$ , the following holds.*

- (1)  $\theta_v$  is injective.
- (2)  $\mathbf{a}$  is the albanese map.
- (3)

$$H^2(M_H(v), \mathbb{Z}) = \theta_v(v^\perp) \oplus \mathbf{a}^* H^2(X \times \widehat{X}, \mathbb{Z}). \quad (1.5)$$

Let  $v$  be the Mukai vector in Theorem 1.1. We set  $K_H(v) := \mathbf{a}^{-1}((0, 0))$ . We shall construct an étale covering such that  $\mathbf{a}$  becomes trivial. By Theorem 0.1, it will become a Bogomolov decomposition of  $M_H(v)$ .

Let  $\mathbf{D}(X)$  and  $\mathbf{D}(\widehat{X})$  be the derived categories of  $X$  and  $\widehat{X}$  respectively. Let  $\mathcal{S} : \mathbf{D}(X) \rightarrow \mathbf{D}(\widehat{X})$  be the Fourier-Mukai transform in [Mu4], that is,  $\mathcal{S}(F) := \mathbf{R}p_{\widehat{X}*}(\mathcal{P} \otimes F)$ ,  $F \in \mathbf{D}(X)$ . Then  $\alpha(E) = \det \mathcal{S}(E) \otimes (\det \mathcal{S}(E_0))^{-1}$ . For a line bundle  $L$  on  $X$ , we set  $\widehat{L} := \det(\mathcal{S}(L))$ . Then the following relations hold.

**Lemma 1.2.**

$$\begin{aligned} \phi_{\widehat{L}} \circ \phi_L &= -\chi(L)1_X, \\ \phi_L \circ \phi_{\widehat{L}} &= -\chi(L)1_{\widehat{X}}. \end{aligned} \quad (1.6)$$

*Proof.* By [Mu4, Prop. 1.21],  $c_1(\widehat{L}) = c_1((-1)^*L) = c_1(L)$  and  $(c_1(\widehat{L}))^2 = (c_1(L))^2$ . So it is sufficient to prove the first equality. By [Mu1, (3.1)], we see that  $\mathcal{S}(L) \otimes \mathcal{P}_x = \mathcal{S}(T_{-x}^*L) = \mathcal{S}(L \otimes \mathcal{P}_{\phi_L(-x)}) = T_{\phi_L(-x)}^*(\mathcal{S}(L))$ . Hence we get that  $\widehat{L} \otimes \mathcal{P}_{\chi(L)x} = T_{\phi_L(-x)}^*(\widehat{L}) = \widehat{L} \otimes \mathcal{P}_{\phi_{\widehat{L}} \circ \phi_L(-x)}$ . Therefore the first equality holds.  $\square$

We define a morphism  $\Phi : K_H(v) \times X \times \widehat{X} \rightarrow M_H(v)$  by  $\Phi(E, x, y) := T_x^*(E) \otimes \mathcal{P}_y$ .

**Lemma 1.3.** *Let  $L$  be a line bundle on  $X$  such that  $c_1(L) = c_1$ . Then,*

$$\begin{aligned} \alpha(T_x^*(E) \otimes \mathcal{P}_y) &= -ax + \phi_{\widehat{L}}(y) \\ \det(T_x^*(E) \otimes \mathcal{P}_y) &= \phi_L(x) + ry. \end{aligned} \quad (1.7)$$

*Proof.* We shall only prove the first equality. By [Mu1, (3.1)], we see that  $\mathcal{S}(T_x^*(E) \otimes \mathcal{P}_y) = T_y^*(\mathcal{S}(T_x^*(E))) = T_y^*(\mathcal{S}(E) \otimes \mathcal{P}_{-x}) = T_y^*(\mathcal{S}(E)) \otimes \mathcal{P}_{-x}$ . Hence  $\det(\mathcal{S}(T_x^*(E) \otimes \mathcal{P}_y)) = T_y^*(\det \mathcal{S}(E)) \otimes \mathcal{P}_{-\chi(E)x}$ . Since  $c_1(\mathcal{S}(E)) = c_1(\mathcal{S}(L))$ ,  $\alpha(T_x^*(E) \otimes \mathcal{P}_y) = \phi_{\det \mathcal{S}(E)}(y) - \chi(E)x = \phi_{\widehat{L}}(y) - \chi(E)x$ . By (1.1),  $\chi(E) = -\langle v(\mathcal{O}_X), v(E) \rangle = a$ , and hence we get the first equality.  $\square$

Let  $\tau : X \times \widehat{X} \rightarrow X \times \widehat{X}$  be a homomorphism sending  $(x, y)$  to  $(rx - \phi_{\widehat{L}}(y), -\phi_L(x) - ay)$ . By Lemma 1.3,  $\mathbf{a} \circ \Phi \circ (1_{K_H(v)} \times \tau)(E, x, y) = (nx, ny)$ , where  $n = \langle v^2 \rangle / 2$ . Let  $\nu : X \times \widehat{X} \rightarrow X \times \widehat{X}$  be the  $n$  times map and we shall consider the fiber product

$$\begin{array}{ccc} M_H(v) \times_{X \times \widehat{X}} X \times \widehat{X} & \longrightarrow & M_H(v) \\ \downarrow & & \downarrow \mathbf{a} \\ X \times \widehat{X} & \xrightarrow{\nu} & X \times \widehat{X} \end{array} \quad (1.8)$$

Then  $\Phi \circ (1_{K_H(v)} \times \tau)$  and the projection  $K_H(v) \times X \times \widehat{X} \rightarrow X \times \widehat{X}$  defines a morphism to the fiber product. We can easily show that this morphism is injective, and hence it is an isomorphism.

*Remark 1.1.* If  $(c_1^2)/2$  and  $r$  are relatively prime, then [Y2, Prop. 4.1] implies that  $M_H(v) \cong \widehat{X} \times \det^{-1}(0)$ . We shall consider the pull-back of  $\mathbf{a} : M_H(v) \rightarrow X \times \widehat{X}$  by the morphism sending  $(x, y)$  to  $(nx, y)$ . Then we get  $M_H(v) \times_{X \times \widehat{X}} X \times \widehat{X} \cong K_H(v) \times X \times \widehat{X}$ .

For simplicity, we also denote the homomorphism  $v^\perp \rightarrow H^2(M_H(v), \mathbb{Z}) \rightarrow H^2(K_H(v), \mathbb{Z})$  by  $\theta_v$ :

$$\theta_v(x) = -\frac{1}{\rho} [p_{K_H(v)*}(\text{ch}(\mathcal{E}_{|K_H(v) \times X})x^\vee)]_1. \quad (1.9)$$

**1.3. Beauville's bilinear form.** Let  $M$  be an irreducible symplectic manifold of dimension  $n$ . Beauville [B] constructed a primitive symmetric bilinear form

$$B_M : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \rightarrow \mathbb{Z}. \quad (1.10)$$

Up to multiplication by positive constants,  $q_M(x) := B_M(x, x)$  satisfies that

$$q_M(x) = \frac{n}{2} \int_M \phi^{n-1} \bar{\phi}^{n-1} x^2 + (1-n) \int_M \phi^n \bar{\phi}^{n-1} x \int_M \phi^{n-1} \bar{\phi}^n x, \quad (1.11)$$

where  $\phi$  is a holomorphic 2 form with  $\int_M \phi^n \bar{\phi}^n = 1$ . For  $\lambda, x \in H^2(M, \mathbb{C})$ , the following relation holds [B, Thm. 5].

$$v(\lambda)^2 q_M(x) = q_M(\lambda) \left[ (2n-1)v(\lambda) \int_M \lambda^{2n-2} x^2 - (2n-2) \left( \int_M \lambda^{2n-1} x \right)^2 \right], \quad (1.12)$$

where  $v(\lambda) = \int_M \lambda^{2n}$ .

## 2. PROOF OF THEOREM 0.1

**2.1. Generalized Kummer variety.** In this subsection, we shall recall Beauville's results [B] on generalized Kummer varieties. Then Theorem 0.1 for  $r = 1$  follows from his results and simple calculations. Let  $X$  be an abelian surface. Let  $\pi : X^n \rightarrow X^{(n)}$  be the  $n$ -th symmetric product of  $X$ . We set  $X^{[n]} := \text{Hilb}_X^n$ . Let  $\gamma : X^{[n]} \rightarrow X^{(n)}$  be the Hilbert-Chow morphism. Let  $\sigma : X^{(n)} \rightarrow X$  be the morphism sending  $(x_1, x_2, \dots, x_n) \in X^{(n)}$  to  $\sum_{i=1}^n x_i \in X$ . Then  $\alpha : X^{[n]} \rightarrow X^{(n)} \rightarrow X$  is the albanese map of  $X^{[n]}$ . If  $n = 2$ , then  $\alpha^{-1}(0)$  is the Kummer surface associated to  $X$  and if  $n \geq 3$ , then  $K_{n-1} := \alpha^{-1}(0)$  is the generalized Kummer variety constructed by Beauville [B].  $K_{n-1}$  is an irreducible symplectic manifold of dimension  $2(n-1)$ .

We assume that  $n \geq 3$ . For integers  $i, j, k$ , we set  $\Delta^{i,j} := \{(x_1, x_2, \dots, x_n) \in X^n | x_i = x_j\}$ ,  $\Delta^{i,j,k} := \Delta^{i,j} \cap \Delta^{j,k}$ . We set  $X_*^n := X^n \setminus \cup_{i < j < k} \Delta^{i,j,k}$ ,  $X_*^{[n]} := X^{[n]} \setminus \cup_{i < j < k} \gamma^{-1}(\pi(\Delta^{i,j,k}))$ . We set  $N := \{(x_1, x_2, \dots, x_n) | x_1 + x_2 + \dots + x_n = 0\}$ ,  $N_* := N \cap X_*^n$ ,  $(K_{n-1})_* := K_{n-1} \cap X_*^{[n]}$ ,  $\delta^{i,j} := \Delta^{i,j} \cap N$ . Since  $n \geq 3$ ,  $\delta^{i,j}$  is connected, indeed, it is isomorphic to  $X^{n-2}$ . Let  $\beta : B_\Delta(X_*^n) \rightarrow X_*^n$  be the blow-up of  $X_*^n$  along  $\delta := \cup_{i < j} \Delta^{i,j}$  and set  $B_\delta(N_*) = \beta^{-1}(N_*)$ . Let  $E^{i,j} := \beta^{-1}(\Delta^{i,j})$  be the exceptional divisor of  $\beta$  and  $e^{i,j} := \beta^{-1}(\Delta^{i,j} \cap N)$ .

$$\begin{array}{ccccc} B_\delta(N_*) & \longrightarrow & B_\Delta(X_*^n) & \xrightarrow{\beta} & X^n \\ \downarrow \pi' & & \downarrow \pi' & & \downarrow \pi \\ (K_{n-1})_* & \longrightarrow & X_*^{[n]} & \xrightarrow{\gamma} & X^{(n)} \\ & & & & \downarrow \sigma \\ & & & & X \end{array} \quad (2.1)$$

We shall first describe  $H^2(K_{n-1}, \mathbb{Z})$ .

**Lemma 2.1.**

$$H^2(K_{n-1}, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \oplus \mathbb{Z}e.$$

*Proof.* By [B, Prop. 8],  $H^2(K_{n-1}, \mathbb{Q}) \cong H^2(X, \mathbb{Q}) \oplus \mathbb{Q}e$ . Since  $\varphi : H^2(X, \mathbb{Z}) \rightarrow H^2(K_{n-1}, \mathbb{Z}) \rightarrow H^2(B_\delta(N_*), \mathbb{Z})$  is injective and  $\text{im } \varphi \subset \beta^*(H^2(N, \mathbb{Z}))$ , we shall prove that the image of  $f : H^2(X, \mathbb{Z}) \rightarrow H^2(N, \mathbb{Z})^{\oplus n}$  is a primitive submodule of  $H^2(N, \mathbb{Z})$ . Let  $\phi : X \times X \rightarrow N$  be the morphism such that  $\phi((x, y)) = (x, y, 0, \dots, 0, -x - y) \in N$ . We shall consider the composition  $g : H^2(X, \mathbb{Z}) \rightarrow H^2(X \times X, \mathbb{Z})$ . Let  $\alpha_i \wedge \alpha_j, i < j$  be the basis of  $H^2(X, \mathbb{Z}) = \wedge^2 H^1(X, \mathbb{Z})$ . Then we see that  $g^*(\alpha_i \wedge \alpha_j) = 2p_1^*(\alpha_i \wedge \alpha_j) + 2p_2^*(\alpha_i \wedge \alpha_j) + (p_1^* \alpha_i \wedge p_2^* \alpha_j - p_1^* \alpha_j \wedge p_2^* \alpha_i)$ . Hence  $\text{im } g$  is a primitive subspace of  $H^2(X \times X, \mathbb{Z})$ . Therefore  $\text{im } f$  is primitive.  $\square$

We shall prove that  $\theta_v$  preserves the bilinear forms. Since  $v = 1 - n\omega$ , we get that  $v^\perp = H^2(X, \mathbb{Z}) \oplus \mathbb{Z}(1 + n\omega)$ . For  $\alpha = x + k(1 + n\omega), x \in H^2(X, \mathbb{Z})$ , simple calculations show that

$$\begin{aligned} \langle \alpha^2 \rangle &= (x^2) - k^2(2n), \\ \theta_v(\alpha) &= \sum_i p_i^*(x) + ke. \end{aligned} \quad (2.2)$$

Hence we shall prove that

$$q_{K_{n-1}}(\theta_v(\alpha)) = (x^2) - k^2(2n). \quad (2.3)$$

We shall choose  $l \in H^2(X, \mathbb{Z})$  with  $(l^2) \neq 0$ . We set  $\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$ . Then the  $\oplus_{i < j} p_i^* H^{ev}(X, \mathbb{Z}) \otimes p_j^* H^{ev}(X, \mathbb{Z})$ -component of  $\sigma^*(\omega) = \sum_{i,j,k,m} p_i^* \alpha_1 \wedge p_j^* \alpha_2 \wedge p_k^* \alpha_3 \wedge p_m^* \alpha_4$  is  $\sum_i p_i^* (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4) + \sum_{i \neq j} p_i^* (\alpha_1 \wedge \alpha_2) \wedge p_j^* (\alpha_3 \wedge \alpha_4)$ . Since the  $\oplus_{i < j} p_i^* H^{ev}(X, \mathbb{Z}) \otimes p_j^* H^{ev}(X, \mathbb{Z})$ -component of  $\sum_{i < j} (\Delta^{i,j} - p_i^* \omega - p_j^* \omega)$  is  $\sum_{i \neq j} p_i^* (\alpha_1 \wedge \alpha_2) \wedge p_j^* (\alpha_3 \wedge \alpha_4)$ , those of  $\mu := \sum_i p_i^* \omega + \sum_{i < j} (\Delta^{i,j} - p_i^* \omega - p_j^* \omega)$  and  $\sigma^*(\omega)$  are the same. Hence we see that

$$\begin{aligned} \int_{K_{n-1}} (\theta_v(l))^{2n-4} (\theta_v(x))^2 &= \frac{1}{n!} \int_N (p_1^* l + \cdots + p_n^* l)^{2n-4} (p_1^* x + \cdots + p_n^* x)^2 \\ &= \frac{1}{n!} \int_{X^n} (p_1^* l + \cdots + p_n^* l)^{2n-4} (p_1^* x + \cdots + p_n^* x)^2 \mu \\ &= \frac{1}{n!} \left\{ \frac{n(n-1)}{2} \int_{X^{n-1}} (2p_1^* l + p_2^* l + \cdots + p_n^* l)^{2n-4} (2p_1^* x + p_2^* x + \cdots + p_n^* x)^2 \right. \\ &\quad \left. - n(n-2) \int_{X^{n-1}} (p_1^* l + p_2^* l + \cdots + p_n^* l)^{2n-4} (p_1^* x + p_2^* x + \cdots + p_n^* x)^2 \right\} \\ &= \frac{(2n-2)!n^2}{n!2^{n-1}} \left( \frac{1}{2n-3} (l^2)^{n-2} (x^2) + \frac{2n-4}{2n-3} (l^2)^{n-2} (l, x)^2 \right). \end{aligned} \quad (2.4)$$

In the same way, we see that

$$\begin{aligned} \int_{K_{n-1}} (\theta_v(l))^{2n-3} (\theta_v(x)) &= \frac{(2n-2)!n^2}{2^{n-1}n!} (l^2)^{n-1} (l, x), \\ \int_{K_{n-1}} (\theta_v(l))^{2n-2} &= \frac{(2n-2)!n^2}{2^{n-1}n!} (l^2)^n. \end{aligned} \quad (2.5)$$

By (1.12), we obtain that

$$q_{K_{n-1}}(\theta_v(x)) = \frac{(x^2)}{(l^2)} q_{K_{n-1}}(\theta_v(l)). \quad (2.6)$$

We shall next compute  $q_{K_{n-1}}(e)$ . We note that  $\gamma^{-1}(\pi(\Delta^{1,2})) = 2e$ . Let  $\iota : H^2(X, \mathbb{Z}) \rightarrow H^2(X^{(n)}, \mathbb{Z})$  be the homomorphism such that  $\pi^*(\iota(x)) = \sum_i p_i^* x \in H^2(X^n, \mathbb{Z})$ . Since the Poincaré dual of  $(\iota(l))^{2n-4} \iota(x)$ ,  $x \in H^2(X, \mathbb{Z})$  (resp.  $(\iota(l))^{2n-4}$ ) is a 2 cycle (resp. 4 cycle) of  $X^{(n)}$ , the intersection with  $\pi(\Delta_{1,2})$  is 0 (resp. finite points).

Let  $\sigma' : X^{n-1} = \Delta_{1,n} \rightarrow X$  be the restriction of  $\sigma$  to the diagonal  $\Delta_{1,n}$ . We set  $\mu' = 2^4 p_1^*(\omega) + \sum_{i=2}^{n-1} p_i^*(\omega) + \sum_{1 < i < j \leq n-1} (\Delta^{i,j} - p_i^*(\omega) - p_j^*(\omega)) + 4 \sum_{i=2}^{n-1} (\Delta_{1,i} - p_1^* \omega - p_i^* \omega)$ . In the same way, we see that the  $\oplus_{i < j} p_i^* H^{ev}(X, \mathbb{Z}) \otimes p_j^* H^{ev}(X, \mathbb{Z})$ -components of  $\sigma^*(\omega)$  and  $\mu'$  are the same. Since  $E_{i,j}$  is the exceptional divisor of  $\beta$ ,  $\beta_*((E_{i,j})|_{E_{i,j}}) = -\Delta^{i,j}$ . Hence we see that

$$\begin{aligned} \int_{K_{n-1}} (\theta_v(l))^{2n-4} \theta_v(x) e &= 0, \\ \int_{K_{n-1}} (\theta_v(l))^{2n-4} e^2 &= -\frac{1}{n!} \int_N (p_1^* l + \cdots + p_n^* l)^{2n-4} \left( \sum_{i < j} \Delta^{i,j} \right) \\ &= -\frac{n(n-1)}{2n!} \int_{X^{n-1}} (2p_1^* l + p_2^* l + \cdots + p_{n-1}^* l)^{2n-4} \mu' \\ &= \frac{(2n-2)!n^2}{2^{n-1}n!} \frac{(-2n)}{2n-3}. \end{aligned} \quad (2.7)$$

Thus  $e$  is orthogonal to  $H^2(X, \mathbb{Z})$  and

$$q_{K_{n-1}}(e) = \frac{-2n}{(l^2)} q_{K_{n-1}}(\theta_v(l)). \quad (2.8)$$

By (2.6) and (2.8), we get (2.3).

**Proposition 2.2** (Beauville). *For  $v = 1 - n\omega$ ,  $n \geq 2$ ,*

$$\theta_v : v^\perp \rightarrow H^2(K_H(v), \mathbb{Z}) \quad (2.9)$$

*is an isometry of Hodge structures.*

**2.2. General cases.** We shall first treat the case where  $X$  is a product of two elliptic curves, and by deformation arguments, we shall treat general cases. Let  $X$  be an abelian surface which is a product of two elliptic curves  $C_1, C_2$ . Let  $f_i$ ,  $i = 1, 2$  be the ample generator of  $H^2(C_i, \mathbb{Z})$ . Let  $(r, d)$  and  $(r_1, d_1)$  be pairs of integers such that  $r > r_1 > 0$  and  $dr_1 - rd_1 = 1$ . We set  $v = r + df_2 - (r - r_1)nf_1 - (d - d_1)n\omega$ . We shall choose an ample divisor  $H = f_2 + mf_1$ ,  $m \gg 0$ . In [Y2, sect. 3.2], we constructed an immersion  $B_\Delta(X^n)/\mathfrak{S}_n \rightarrow M_H(v)$  which is an isomorphism in codimension 1. Hence we get an isomorphism  $H^2(M_H(v), \mathbb{Z}) \rightarrow H^2(B_\Delta(X^n), \mathbb{Z})^{\mathfrak{S}_n}$ . It also induces a birational map  $(K_{n-1})_* \rightarrow K_H(v)$  which is an isomorphism in codimension 1. Thus we get an isomorphism  $H^2(K_H(v), \mathbb{Z}) \rightarrow H^2(K_{n-1}, \mathbb{Z})$ . By the injective homomorphism  $H^2(K_H(v), \mathbb{Z}) \rightarrow H^2(B_\delta(N_*), \mathbb{Z})$ , we shall regard  $H^2(K_H(v), \mathbb{Z})$  as a submodule of  $H^2(B_\delta(N_*), \mathbb{Z})$ .

We set  $x = x_1 + x_2f_1 + x_3f_2 + x_4\omega + D$ ,  $D \in H^1(C_1, \mathbb{Z}) \otimes H^1(C_2, \mathbb{Z})$ . We assume that  $r \geq 2r_1$ . In the notation of [Y2, sect. 3.1],  $\theta_v(x) = -\kappa_2(x^\vee)$ . Hence [Y2, (3.19), (3.20)] are written down as follows:

$$\theta_v(x) = y_1\left(\sum_{i=1}^n p_i^* f_2\right) + y_2\left(\sum_{i=1}^n p_i^* f_1\right) + y_3\left(\sum_{i < j} E^{i,j}\right) + \sum_{i=1}^n p_i^* D, \quad (2.10)$$

where

$$\begin{cases} y_1 = dx_1 - rx_3 \\ y_2 = -(d - d_1)x_2 + (r - r_1)x_4 - n((d - 2d_1)x_1 - (r - 2r_1)x_3) \\ y_3 = -d_1x_1 + r_1x_3 \\ y_4 = dx_2 - rx_4 + n((d - d_1)x_1 - (r - r_1)x_3). \end{cases} \quad (2.11)$$

By simple calculations, we get that

$$\begin{cases} x_1 = r_1y_1 + ry_3 \\ x_2 = -nr_1y_1 - ry_2 - n(r + r_1)y_3 - (r - r_1)y_4 \\ x_3 = d_1y_1 + dy_3 \\ x_4 = -nd_1y_1 - dy_2 - n(d + d_1)y_1 - d_2y_4. \end{cases} \quad (2.12)$$

By the definition of  $v^\perp$ ,  $x$  belongs to  $v^\perp$  if and only if  $y_4 = 0$ . Hence we obtain that

$$\begin{aligned} \langle x^2 \rangle &= 2x_2x_3 - 2x_1x_4 + (D^2) \\ &= 2y_1y_2 + (D^2) - 2ny_3^2 \\ &= q_{K_H(v)}(\theta_v(x)). \end{aligned} \quad (2.13)$$

**Lemma 2.3.** *Under the same assumptions on  $X$ , let  $v = r + (df_2 + sf_1) + a\omega$  be a Mukai vector such that  $(r, d) = 1$  and  $\langle v^2 \rangle = 2n \geq 4$ . We set  $H := f_2 + mf_1$ , where  $m$  is a sufficiently large integer. Then  $K_H(v)$  is an irreducible symplectic manifold and*

$$\theta_v : v^\perp \rightarrow H^2(K_H(v), \mathbb{Z}) \quad (2.14)$$

*is an isometry of Hodge structures for  $n \geq 3$ .*

*Proof.* We shall choose a pair of integers  $(r_1, d_1)$  such that  $r > r_1 > 0$  and  $dr_1 - rd_1 = 1$ . We first assume that  $r \geq 2r_1$ . Since  $\langle v^2 \rangle = 2(ds - ra) = 2n$ , there are integers  $s_1, a_1$  such that

$$\begin{cases} s = nr_1 + s_1r, \\ a = nd_1 + a_1d. \end{cases} \quad (2.15)$$

We set  $v' := v \text{ch}(\mathcal{O}_X(-(n + s_1)f_1))$ . Then we see that  $v' = r + df_2 - (r - r_1)nf_1 - (d - d_1)n\omega$ . Since  $K_H(v')$  is an irreducible symplectic manifold and  $\theta_{v'}$  is an isometry, combining (1.4), the assertions hold for  $K_H(v)$  with  $r \geq 2r_1$ . If  $r < 2r_1$ , then we shall replace  $v$  by  $v^\vee$ . Since  $\theta_{v^\vee}(x^\vee) = -\theta_v(x)$ ,  $x \in H^{ev}(X, \mathbb{Z})$ , this case can be reduced to the first case.  $\square$

We shall treat general cases. Twisting by some ample line bundles, we may assume that  $\xi$  belongs to the ample cone. The following argument is the same as that in [Y2, Prop. 3.3]. Let  $f : (\mathcal{X}, \mathcal{L}) \rightarrow T$  be a family of polarized abelian surfaces over a connected curve  $T$  such that  $\text{NS}(\mathcal{X}_t) = \mathbb{Z}\mathcal{L}_t$  for some  $t \in T$ . Let  $v := r + d\mathcal{L} + a\omega \in R^{ev}f_*\mathbb{Z}$  be a family of Mukai vector such that  $(r, d) = 1$ . By [Y2, Prop. 3.3], we can construct a proper and smooth family of moduli spaces  $\mathcal{M}_{\mathcal{X}/T}(v) \rightarrow T$  and a family of albanese maps  $\mathbf{a}_T : \mathcal{M}_{\mathcal{X}/T}(v) \rightarrow \mathcal{X} \times_T \text{Pic}_{\mathcal{X}/T}^0$ . Since  $\mathcal{X} \rightarrow T$  is projective, we can also construct a family of homomorphisms  $(\theta_v)_t : (v^\perp)_t \rightarrow H^2((\mathcal{M}_{\mathcal{X}/T}(v))_t, \mathbb{Z})$ . Let  $0_T : T \rightarrow \mathcal{X} \times_T \text{Pic}_{\mathcal{X}/T}^0$  be the 0-section of  $f$ , and we set  $\mathcal{K}_{\mathcal{X}/T}(v) := \mathbf{a}_T^{-1}(0_T)$ . We assume that  $(\mathcal{K}_{\mathcal{X}/T}(v))_{t_0}, t_0 \in T$  is irreducible symplectic and  $(\theta(v))_{t_0} : (v^\perp)_{t_0} \rightarrow H^2((\mathcal{K}_{\mathcal{X}/T}(v))_{t_0}, \mathbb{Z})$  is an isometry of Hodge structures. Then every fiber of  $\mathbf{a}$  is irreducible symplectic and  $(\theta_v)_t$  is an isometry of Hodge structures. We note that moduli of  $(1, n)$ -polarized abelian surfaces is irreducible. Applying this assertion, we can reduce to the case where  $X$  is a product of elliptic curves. Applying Lemma 2.3, we get Theorem 0.1.  $\square$

**Corollary 2.4.** *We set  $(v^\perp)_{alg} := v^\perp \cap (H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}))$ . Then  $\theta_v$  induces an isometry*

$$(v^\perp)_{alg} \rightarrow \text{NS}(K_H(v)).$$

The following example is similar to [Mu5, 5.17].

*Example 1.* Let  $X$  be an abelian surface with  $\text{NS}(X) = \mathbb{Z}H$ ,  $(H^2) = 2$ . We set  $v = 2 + H - 2\omega$ . Then  $M_H(v)$  is a variety of dimension 12. It is easy to see that  $v^\perp$  is generated by  $\alpha := 1 + \omega$  and  $\beta = H + \omega$ . Since

$$\langle \alpha^2 \rangle = -2, \quad \langle \alpha, \beta \rangle = -1, \quad \langle \beta^2 \rangle = 2, \quad (2.16)$$

$\text{NS}(K_H(v))$  is indecomposable. Hence  $M_H(v)$  is not birationally equivalent to  $\hat{Y} \times \text{Hilb}_Y^5$  for any  $Y$ .

### 3. THE CASE OF $\langle v^2 \rangle = 4$

In this section, we shall treat the remaining case, that is,  $\langle v^2 \rangle = 4$ . In this case,  $K_H(v)$  is a K3 surface. We shall determine this K3 surface. Let  $v = r + \xi + a\omega$ ,  $\xi \in H^2(X, \mathbb{Z})$  be a Mukai vector such that  $r + \xi$  is primitive and  $\langle v^2 \rangle = 4$ . Replacing  $v$  by  $v \text{ch}(H^{\otimes m})$ ,  $m \gg 0$ , we may assume that  $\xi$  belongs to the ample cone. Let  $\iota : X \rightarrow X$  be the  $(-1)$ -involution of  $X$  and  $x_1, x_2, \dots, x_{16}$  the fixed points of  $\iota$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blow-ups of  $X$  at  $x_1, x_2, \dots, x_{16}$  and  $E_1, E_2, \dots, E_{16}$  the exceptional divisors of  $\pi$ . Let  $q_1 : X \rightarrow X/\iota$  be the quotient map. Then the morphism  $q_1 \circ \pi : \tilde{X} \rightarrow X/\iota$  factors through the quotient  $\tilde{X}/\iota$  of  $\tilde{X}$  by  $\iota$  :  $\tilde{X} \xrightarrow{q_2} \tilde{X}/\iota \xrightarrow{\varpi} X/\iota$ .  $\text{Km}(X) := \tilde{X}/\iota$  is the Kummer surface associated to  $X$  and  $\varpi : \tilde{X}/\iota \rightarrow X/\iota$  is the minimal resolution of  $X/\iota$ . We set  $C_i := q_2(E_i)$ ,  $i = 1, 2, \dots, 16$ .

We may assume that  $H$  is symmetric, that is,  $\iota^*H = H$ . Then  $H$  has a  $\iota$ -linearization. Hence  $H^{\otimes 2}$  descent to an ample line bundle  $L$  on  $X/\iota$ . Then  $L_m := \varpi^*(L^{\otimes m})(-\sum_{i=1}^{16} C_i)$ ,  $m \gg 0$  is an ample line bundle on  $\text{Km}(X)$ . We shall fix a sufficiently large integer  $m$ . Let  $w = r + c_1 + b\omega \in H^{ev}(\text{Km}(X), \mathbb{Z})$  be an isotropic Mukai vector. We shall consider moduli space  $M_{L_m}(w)$ . By Mukai [Mu3],  $M_{L_m}(w)$  is not empty. By our assumption on  $L_m$ ,  $M_{L_m}(w)$  consists of  $\mu$ -stable sheaves. Indeed, for a  $\mu$ -semi-stable vector bundle  $F$  of  $v(F) = w + k\omega$ ,  $k \geq 0$ ,  $q_2^*(F)$  is a  $\mu$ -semi-stable vector bundle on  $X$  with respect to  $\pi^*(H^{\otimes 2m})(-2\sum_{i=1}^{16} E_i)$ . Since  $m$  is sufficiently large and  $H$  is a general ample line bundle,  $q_2^*(F)$  is a  $\mu$ -stable vector bundle. Hence  $F$  is a  $\mu$ -stable vector bundle, which implies that  $M_{L_m}(w)$  consists of  $\mu$ -stable sheaves. Since  $\dim M_{L_m}(w) = \langle w^2 \rangle + 2 = 2$ , every member of  $M_{L_m}(w)$  is locally free. Moreover general members  $F$  of  $M_{L_m}(w)$  are rigid on each  $(-2)$ -curves  $C_i$ .

**Lemma 3.1.** *We set  $N(w, i) := \{F \in M_{L_m}(w) | F|_{C_i} \text{ is not rigid}\}$ . Then  $N(w, i)$  is not empty if and only if  $r | \deg(F|_{C_i})$ . Moreover if  $N(w, i)$  is not empty, then  $N(w, i)$  is a rational curve.*

*Proof.* We assume that  $F|_{C_i}$  is not rigid. We set  $F|_{C_i} = \bigoplus_{j=1}^k \mathcal{O}_{C_i}(a_j)^{\oplus n_j}$ ,  $a_1 < a_2 < \dots < a_k$ . Let  $F' := \ker(F \rightarrow \mathcal{O}_{C_i}(a_1)^{\oplus n_1})$  be the elementary transformation of  $F$  along  $\mathcal{O}_{C_i}(a_1)^{\oplus n_1}$ . Then  $v(F') = v(F) - n_1(C_i - (a_1 + 1)\omega)$ . Hence we see that  $\langle v(F')^2 \rangle = -2n_1(\sum_{j \geq 2} n_j(a_j - a_1 - 1))$ . By the choice of  $L_m$ ,  $F'$  is also  $\mu$ -stable. Hence  $-2 \leq -2n_1(\sum_{j \geq 2} n_j(a_j - a_1 - 1))$ . Since  $F$  is not rigid,  $\sum_{j \geq 2} n_j(a_j - a_1 - 1) > 0$ . Thus  $n_1 = \sum_{j \geq 2} n_j(a_j - a_1 - 1) = 1$ . Therefore we get that  $F|_{C_i} \cong \mathcal{O}_{C_i}(a_1) \oplus \mathcal{O}_{C_i}(a_1 + 1)^{\oplus (r-2)} \oplus \mathcal{O}_{C_i}(a_1 + 2)$ . In this case,  $\langle v(F')^2 \rangle = -2$ , and hence  $F'$  is a unique stable vector bundle of  $v(F') = v(F) - n_1(C_i - (a_1 + 1)\omega)$ . It is not difficult to see that the choice of inverse transformations is parametrized by  $\mathbb{P}^1$ . Therefore  $N(w, i)$  is a rational curve.  $\square$

We shall consider the pull-back  $q_2^*(F)$  of a general member  $F$ . Since  $F|_{C_i}$ ,  $1 \leq i \leq 16$  are rigid, replacing  $q_2^*(F)$  by  $q_2^*(F)(\sum_{i=1}^{16} s_i E_i)$ , we may assume that  $q_2^*(F)|_{E_i} \cong \mathcal{O}_{E_i}(-1)^{\oplus k_i} \oplus \mathcal{O}_{E_i}^{\oplus(r-k_i)}$ . Let  $\phi : q_2^*(F) \rightarrow \oplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus k_i}$  be the quotient map induced by the quotients  $q_2^*(F)|_{E_i} \rightarrow \mathcal{O}_{E_i}(-1)^{\oplus k_i}$ . Then  $G := \ker \phi$  is the elementary transformation of  $q_2^*(F)$  along  $\oplus_{i=1}^{16} \mathcal{O}_{E_i}(-1)^{\oplus k_i}$  and  $G$  satisfies that  $G|_{E_i} \cong \mathcal{O}_{E_i}^{\oplus r}$ . Hence  $\pi_*(G)$  is a stable vector bundle on  $X$ . So we get a rational map  $f : M_{L_m}(w) \cdots \rightarrow M_H(v)$ , where  $v = v(\pi_*(G))$ . Since  $M_{L_m}(w)$  is a K3 surface, the image of  $M_{L_m}(w)$  belongs to a fiber of  $\mathfrak{a}$ . Since  $q_2^*(F)$  is a stable, and hence a simple vector bundle and  $\iota$  has fixed points,  $\iota$ -linearization on  $F$  is uniquely determined by  $q_2^*(F)$ . Hence  $f$  is generically injective. By a simple calculation, we get that

$$\begin{aligned} \langle v(G)^2 \rangle &= 2rc_2(G) - (r-1)(c_1(G)^2) \\ &= 4rc_2(F) - 2(r-1)(c_1(F)^2) - \sum_{i=1}^{16} k_i(r-k_i) \\ &= 2(\langle w^2 \rangle + 2r^2) - \sum_{i=1}^{16} k_i(r-k_i) \\ &= 4r^2 - \sum_{i=1}^{16} k_i(r-k_i). \end{aligned} \tag{3.1}$$

Hence if  $\langle v(G)^2 \rangle = 4$ , then the fiber of  $\mathfrak{a}$  is isomorphic to  $M_{L_m}(w)$ .

Conversely for a Mukai vector  $v = r + dN + a\omega \in H^{ev}(X, \mathbb{Z})$  such that (a)  $N$  is a  $(1, n)$ -polarization, (b)  $(r, d) = 1$  and (c)  $\langle v^2 \rangle = d^2(N^2) - 2ra = 4$ , we shall look for such a vector  $w \in H^{ev}(\text{Km}(X), \mathbb{Z})$ . We shall divide the problem into two cases.

Case (I). We first assume that  $r$  is even. In this case,  $d$  must be odd. By the condition (c),  $(N^2) = 2n$  is divisible by 4. Thus  $n$  is an even integer. In this case, replacing  $N$  by  $N \otimes \mathcal{P}$  with  $\mathcal{P}^{\otimes 2} \cong \mathcal{O}_X$ , we may assume that  $N$  has a  $\iota$ -linearization which acts trivially on the fibers of  $N$  at exactly 4 points (cf. [L-B, Rem. 7.7]). Replacing the indices, we assume that the 4 points are  $x_1, x_2, x_3, x_4$ . We set  $N_1 := \pi^*(N^{\otimes d})(\frac{r-2}{2} \sum_{i=1}^4 E_i + \frac{r}{2} \sum_{i \geq 5} E_i)$  and  $N_2 := N_1(-rE_1)$ . Then for suitable linearizations,  $N_1$  and  $N_2$  descend to line bundles  $\xi_1$  and  $\xi_2$  on  $\text{Km}(X)$  respectively. By simple calculations, we get that

$$\begin{aligned} (\xi_1^2) &= d^2 \frac{(N^2)}{2} - 2r^2 + 2r - 2 \\ &= r(a - 2r + 2), \\ (\xi_2^2) &= r(a - 2r + 1). \end{aligned} \tag{3.2}$$

We set

$$w := \begin{cases} r + \xi_1 + \frac{a-2r+2}{2}\omega, & \text{if } a \text{ is even,} \\ r + \xi_2 + \frac{a-2r+1}{2}\omega, & \text{if } a \text{ is odd.} \end{cases} \tag{3.3}$$

Then we get that  $\langle w^2 \rangle = 0$ . Let  $F$  be a general stable vector bundle of  $v(F) = w$ . By the choice of  $\xi_1$  and  $\xi_2$ , the restriction of  $q_2^*(F)$  or  $q_2^*(F)(E_1)$  to  $E_i$  is isomorphic to  $\mathcal{O}_{E_i}(-1)^{\oplus k_i} \oplus \mathcal{O}_{E_i}^{\oplus(r-k_i)}$ , where  $k_i = (r-2)/2$  for  $1 \leq i \leq 4$  and  $k_i = r/2$  for  $i \geq 5$ . Then by (3.1), we get that  $\langle v(\pi_*(G))^2 \rangle = 4$ . Since  $\text{rk}(\pi_*(G)) = r$  and  $c_1(\pi_*(G)) = dN$ ,  $v(\pi_*(G))$  must be equal to  $v$ . Therefore  $K_H(v)$  is isomorphic to  $M_{L_m}(w)$ .

Case (II). We assume that  $r$  is odd. Replacing  $v$  by  $v \text{ ch}(N)$ , we may assume that  $d$  is even. We set  $N_1 := \pi^*(N^{\otimes d})(\frac{r-1}{2} \sum_{i=1}^{16} E_i)$ . Then for a suitable linearization,  $N_1$  descend to a line bundle  $\xi$  on  $\text{Km}(X)$ . By a simple calculation, we get that  $(\xi^2) = r(a - 2r + 4)$ . Since  $d$  is even and  $r$  is odd, condition (c) implies that  $a$  is an even integer. We set  $v := r + \xi + \{(a - 2r + 4)/2\}\omega$ . Then we get that  $\langle v^2 \rangle = 0$ . In the same way as above, we see that  $v(\pi_*(G)) = v$ , which implies that  $K_H(v) \cong M_{L_m}(w)$ .

**Theorem 3.2.** *Let  $v = r + \xi + a\omega \in H^{ev}(X, \mathbb{Z})$  be a Mukai vector such that  $r > 0$ ,  $r + \xi$  is primitive and  $\langle v^2 \rangle = 4$ . Let  $\text{Km}(X)$  be the Kummer surface associated to  $X$ . Then there is an isotropic Mukai vector  $w \in H^{ev}(\text{Km}(X), \mathbb{Z})$  and an ample line bundle  $H'$  on  $\text{Km}(X)$  such that  $K_H(v)$  is isomorphic to  $M_{H'}(w)$ .*

*Remark 3.1.* By the choice of  $k_i$ , if  $r > 2$ , then  $N(w, i)$  is empty. Thus  $f$  is a morphism. If  $r = 2$ , then  $N(w, i)$ ,  $1 \leq i \leq 4$  is not empty and these closed subset correspond to the closed subset  $N(v, i) := \{G \in K_H(v) | G \text{ is not locally free at } x_i\}$ .

#### 4. APPENDIX

In this appendix, we shall explain another method to prove Theorem 0.1. Let  $(X_1, X_2, \mathcal{P})$  be a triple of surfaces  $X_1, X_2$  and a coherent sheaf  $\mathcal{P}$  on  $X_1 \times X_2$  such that  $K_{X_1}$  and  $K_{X_2}$  are trivial,  $\mathcal{P}$  is flat over  $X_1$  and  $X_2$ , and  $\mathcal{P}$  is strongly simple over  $X_1$  and  $X_2$  (see [Br, sect. 2]). We denote the projections  $X_1 \times X_2 \rightarrow X_i, i = 1, 2$  by  $p_i$ . Let  $\mathcal{F}_D : \mathbf{D}(X_1) \rightarrow \mathbf{D}(X_2)$  be the Fourier-Mukai transform defined by  $\mathcal{P}$ , that is,  $\mathcal{F}_D(x) = \mathbf{R}p_{2*}(\mathcal{P} \otimes p_1^*(x)), x \in \mathbf{D}(X_1)$ . Let  $\widehat{\mathcal{F}}_D : \mathbf{D}(X_2) \rightarrow \mathbf{D}(X_1)$  be the inverse transformation, that is,  $\widehat{\mathcal{F}}_D(y) = \mathbf{R}\mathrm{Hom}_{p_1}(\mathcal{P}, p_2^*(y)), y \in \mathbf{D}(X_2)$ . Let  $\mathcal{F}_H : H^{ev}(X_1, \mathbb{Q}) \rightarrow H^{ev}(X_2, \mathbb{Q})$  and  $\widehat{\mathcal{F}}_H : H^{ev}(X_2, \mathbb{Q}) \rightarrow H^{ev}(X_1, \mathbb{Q})$  be homomorphisms such that

$$\mathcal{F}_H(x) = p_{2*}((\mathrm{ch} \mathcal{P})p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_1^*(x)), x \in H^{ev}(X_1, \mathbb{Q}), \quad (4.1)$$

$$\widehat{\mathcal{F}}_H(y) = p_{1*}((\mathrm{ch} \mathcal{P})^\vee p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_2^*(y)), y \in H^{ev}(X_2, \mathbb{Q}). \quad (4.2)$$

By Grothendieck Riemann-Roch theorem, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{D}(X_1) & \xrightarrow{\mathcal{F}_D} & \mathbf{D}(X_2) \\ \sqrt{\mathrm{td}_{X_1}} \mathrm{ch} \downarrow & & \downarrow \sqrt{\mathrm{td}_{X_2}} \mathrm{ch} \\ H^{ev}(X_1, \mathbb{Q}) & \xrightarrow{\mathcal{F}_H} & H^{ev}(X_2, \mathbb{Q}) \end{array} \quad (4.3)$$

**Lemma 4.1.** *For  $x \in H^{ev}(X_1, \mathbb{Z})$ ,  $y \in H^{ev}(X_2, \mathbb{Z})$ , we get  $\langle \mathcal{F}_H(x), y \rangle = \langle x, \widehat{\mathcal{F}}_H(y) \rangle$ .*

*Proof.*

$$\begin{aligned} \langle \mathcal{F}_H(x), y \rangle &= - \int_{X_2} (p_{2*}((\mathrm{ch} \mathcal{P})p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_1^*(x))y)^\vee \\ &= - \int_{X_1 \times X_2} ((\mathrm{ch} \mathcal{P})p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_1^*(x)p_2^*(y))^\vee \\ &= - \int_{X_1 \times X_2} p_1^*(x)((\mathrm{ch} \mathcal{P})^\vee p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_2^*(y))^\vee \\ &= - \int_{X_1} x \{p_{1*}((\mathrm{ch} \mathcal{P})^\vee p_1^* \sqrt{\mathrm{td}_{X_1}} p_2^* \sqrt{\mathrm{td}_{X_2}} p_2^*(y))\}^\vee \\ &= \langle x, \widehat{\mathcal{F}}_H(y) \rangle. \end{aligned} \quad (4.4)$$

□

**Lemma 4.2.** *For  $x \in H^{ev}(X_1, \mathbb{Z})$ ,  $\mathcal{F}_H(x)$  belongs to  $H^{ev}(X_2, \mathbb{Z})$ . In particular  $\mathcal{F}_H$  is an isometry of Mukai lattice.*

*Proof.* By [Y3, sect. 2],  $[\mathcal{F}(x)]_1$  belongs to  $H^2(X_2, \mathbb{Z})$ . By Lemma 4.1, we get

$$\begin{aligned} \langle \mathcal{F}_H(x), 1 \rangle &= \langle x, \widehat{\mathcal{F}}_H(1) \rangle \in \mathbb{Z}, \\ \langle \mathcal{F}_H(x), \omega_2 \rangle &= \langle x, \widehat{\mathcal{F}}_H(\omega_2) \rangle \in \mathbb{Z}. \end{aligned} \quad (4.5)$$

Hence  $\mathcal{F}(x) \in H^{ev}(X_2, \mathbb{Z})$ . □

Let  $H$  and  $H'$  be ample divisors on  $X_1$  and  $X_2$  respectively, and  $v \in H^{ev}(X, \mathbb{Z})$  a Mukai vector. Let  $U$  be an open subscheme of  $M_H(v)$  such that WIT <sub>$i$</sub>  holds for  $E \in U$  and  $R^i p_{2*}(\mathcal{P} \otimes p_1^*(E))$  belongs to  $M_{H'}(w)$ , where  $w = (-1)^i \mathcal{F}_H(v)$ . We assume that  $\mathrm{codim}_{M_H(v)}(M_H(v) \setminus U) \geq 2$  and  $U \rightarrow M_{H'}(w)$  is birational. We denote the image of  $U$  by  $V$ . We set  $U = X_0$  and we denote projections  $U \times X_1 \times X_2 \rightarrow X_i$  and  $U \times X_1 \times X_2 \rightarrow X_i \times X_j$  by  $q_i$  and  $q_{ij}$  respectively. We also denote the projection  $U \times X_i \rightarrow U$  by  $r_i$  and the projection  $U \times X_i \rightarrow X_i$  by  $s_i$ . Let  $\mathcal{E}$  be a quasi-universal family of similitude  $\rho$  on  $U \times X_1$ . By the identification  $U \rightarrow V$ ,  $R^i q_{02*}(q_{12}^* \mathcal{P} \otimes q_{01}^* \mathcal{E})$  becomes a quasi-universal family of similitude  $\rho$  on  $V \times X_2$ .

**Proposition 4.3.**  *$\mathcal{F}_H$  induces an isometry  $v^\perp \rightarrow w^\perp$  and the following diagram is commutative.*

$$\begin{array}{ccc} v^\perp & \xrightarrow{(-1)^i \mathcal{F}_H} & w^\perp \\ \theta_v \downarrow & & \downarrow \theta_w \\ H^2(K_H(v), \mathbb{Z}) & \xlongequal{\quad} & H^2(K_H(w), \mathbb{Z}) \end{array} \quad (4.6)$$

where  $K_H(v)$  is a fiber of an albanese map  $M_H(v) \rightarrow \text{Alb}(M_H(v))$ . In particular, if  $K_H(v)$  is irreducible symplectic, then  $\theta_v$  is an isometry of Hodge structures if and only if  $\theta_w$  is an isometry of Hodge structures.

*Proof.* The first assertion follows from Lemma 4.1. For  $y \in w^\perp$ , we see that

$$\begin{aligned}
\rho\theta_w(y) &= (-1)^i \left[ r_{2*}(\text{ch}((-1)^i R^i q_{02*}(q_{12*}^*(\mathcal{P}) \otimes q_{01}^*(\mathcal{E}))) p_2^*(\sqrt{\text{td}_{X_2} y^\vee}) \right]_1 \\
&= (-1)^i \left[ r_{2*}(q_{12}^*(\text{ch } \mathcal{P}) q_{01}^*(\text{ch } \mathcal{E}) q_1^*(\text{td}_{X_1}) p_2^*(\sqrt{\text{td}_{X_2} y^\vee})) \right]_1 \\
&= (-1)^i \left[ r_{1*} q_{01*}(q_{01}^*(\text{ch } \mathcal{E}) q_1^*(\sqrt{\text{td}_{X_1}}) q_{12}^*(\text{ch } \mathcal{P}) p_1^*(\sqrt{\text{td}_{X_1}}) p_2^*(\sqrt{\text{td}_{X_2} y^\vee})) \right]_1 \\
&= (-1)^i \left[ r_{1*}((\text{ch } \mathcal{E}) p_1^*(\sqrt{\text{td}_{X_1}}) s_1^*(p_{1*}((\text{ch } \mathcal{P})^\vee p_1^*(\sqrt{\text{td}_{X_1}}) p_2^*(\sqrt{\text{td}_{X_2} y^\vee}))^\vee)) \right]_1 \\
&= (-1)^i \left[ r_{1*}((\text{ch } \mathcal{E}) p_1^*(\sqrt{\text{td}_{X_1}}) s_1^*(\widehat{\mathcal{F}}_H(y))^\vee) \right]_1 \\
&= \rho\theta_v((-1)^i \widehat{\mathcal{F}}_H(y)^\vee).
\end{aligned} \tag{4.7}$$

Since  $\widehat{\mathcal{F}}_H \circ \mathcal{F}_H = 1_{H^{ev}(X_1, \mathbb{Z})}$ , we get (4.6).  $\square$

Let  $\pi : X \rightarrow C$  be an elliptic K3 surface or an elliptic abelian surface. Let  $f$  be a fiber of  $\pi$  and  $\sigma$  is a section of  $\pi$ . We set  $v = r + (\sigma + kf) + a\omega \in H^{ev}(X, \mathbb{Z})$ . We shall choose a polarization  $H = \sigma + nf, n \gg 0$ . By using Fourier-Mukai transformations, Bridgeland [Br] constructed a birational map  $M_H(v) \cdots \rightarrow \text{Pic}^0(X) \times \text{Hilb}_X^m$ , where  $2m + 2 = \dim M_H(v)$ . Moreover if  $r \geq 3$ , then this birational map is defined by Fourier-Mukai Transformation on the complement of a codimension 2 subset of  $M_H(v)$ . So we can apply Proposition 4.3. By deformation arguments which are more complicated than those in 2.2, we can reprove Theorem 0.1 for  $r \geq 3$ .

*Acknowledgement.* I would like to thank Professor T. Katsura for valuable suggestions. I would also like to thank Max Planck Institut für Mathematik for support and hospitality.

## REFERENCES

- [B] Beauville, A., *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755–782
- [Br] Bridgeland, T., *Fourier-Mukai transforms for elliptic surfaces*, J. reine angew. Math. **498** (1998), 115–133
- [G] Gieseker, D., *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. **106** (1977), 45–60
- [G-H] Göttsche, L., Huybrechts, D., *Hodge numbers of moduli spaces of stable bundles on K3 surfaces*, Internat. J. Math. **7** (1996), 359–372
- [L-B] Lange, H., Birkenhake, Ch., *Complex Abelian Varieties*, Springer-Verlag
- [Ma1] Maruyama, M., *Moduli of stable sheaves II*, J. Math. Kyoto Univ. **18** (1978), 557–614
- [Ma2] Maruyama, M., *Moduli of algebraic vector bundles*, in preparation
- [Mu1] Mukai, S., *Duality between  $D(X)$  and  $D(\hat{X})$  with its application to Picard sheaves*, Nagoya Math. J., **81** (1981), 153–175
- [Mu2] Mukai, S., *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. math. **77** (1984), 101–116
- [Mu3] Mukai, S., *On the moduli space of bundles on K3 surfaces I*, Vector bundles on Algebraic Varieties, Oxford, 1987, 341–413
- [Mu4] Mukai, S., *Fourier functor and its application to the moduli of bundles on an Abelian variety*, Adv. Studies in Pure Math. **10** (1987), 515–550
- [Mu5] Mukai, S., *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*, Sugaku Expositions, **1** (1988), 139–174
- [O] O’Grady, K., *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. Algebraic Geom., **6** (1997), no. 4, 599–644
- [Y1] Yoshioka, K., *Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Internat. J. Math. **7** (1996), 411–431
- [Y2] Yoshioka, K., *Some notes on the moduli of stable sheaves on elliptic surfaces*, Nagoya Math. J. to appear
- [Y3] Yoshioka, K., *A note on the universal family of moduli of stable sheaves*, J. reine angew. math. **496** (1998), 149–161
- [Y4] Yoshioka, K., *An application of exceptional bundles to the moduli of stable sheaves on a K3 surface*, alg-geom/9705027, *Some examples of Mukai’s reflections on K3 surfaces*, (extended version)

MAX PLANCK INSTITUT FÜR MATHEMATIK, GOTTFRIED CLAREN STR. 26, D-53225 BONN, GERMANY &  
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KOBE UNIVERSITY, KOBE, 657, JAPAN  
E-mail address: yoshioka@mpim-bonn.mpg.de